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Almost Convergence and Nonlinear Ergodic Theorems

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The first ergodic theorems for nonlinear nonexpansive mappings and semigroups in Hilbert space were established by Baillon [1, 2, 3] and by Baillon and Brézis [4]. Brézis and Browder [6] have recently replaced the Cesàro method in the discrete case by more general summability methods. A similar extension in the continuous case has been obtained in [10]. The purpose of this note is to show how Lorentz's concept of almost convergence [9] can be used to improve these results and to simplify their proofs.

Following Lorentz, we say that a regular matrix $\{a_{n,k}\}$ is strongly regular if $\lim_{n\to\infty} \sum_{k=0}^{\infty} | a_{n,k+1} - a_{n,k} | = 0$, and that a bounded sequence $\{x_n\}$ in a Banach space E is almost (strongly) convergent to $z \in E$ if the strong $\lim_{n\to\infty} (\sum_{i=0}^{n-1} x_{k+i})/n = z$ uniformly in k. We say that $\{x_n\}$ is almost weakly convergent to z if $\{(x_n, y)\}$ is almost convergent to (z, y) for all $y \in E^*$.

THEOREM 1. *Let H be a real Hilbert space, C a closed convex subset of H, $T : C \to C$ a nonexpansive mapping with a fixed point, and $x \in C$. Let $\{a_{n,k}\}$ be strongly regular and $y_n = \sum_{k=0}^{\infty} a_{n,k} T^k x$.*

(a) *$\{y_n\}$ converges weakly to z, a fixed point of T that is the asymptotic center* [8] *of $\{T^n x\}$.*

(b) *If, in addition, $\lim_{n\to\infty} (T^n x, T^{n+i} x)$ exists uniformly in $i \geqslant 0$ (for example if T is odd), then $\{y_n\}$ converges strongly.*

Proof. We slightly modify Baillon's proofs of the Cesàro method results as simplified by Brézis. For any sequence $\{x_n\}$ denote $(\sum_{i=0}^{n-1} x_{k+i})/n$ by $S_n(x_k)$. Let F be the fixed point set of T and $P : C \to F$ the nearest point projection. Denote $T^n x$ by x_n, and let $\{k(n)\}$ be an arbitrary sequence of natural numbers and f any point in F. We note that $\{x_n\}$ is bounded, $\{Px_n\}$ converges strongly to z, and

$$(S_n(Px_{k(n)}) - S_n(x_{k(n)}), f - z) \geqslant -MS_n(| Px_{k(n)} - z |)$$

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for some constant M. We also have

$$|S_n(x_{k(n)}) - TS_n(x_{k(n)})| \leq (1/n^{1/2}) | x_{k(n)} - TS_n(x_{k(n)})| .$$

Therefore if $\{S_{n_j}(x_{k(n_j)})\}$ converges weakly to q, then on the one hand $(z - q, f - z) \geq 0$ for all $f \in F$ and consequently $Pq = z$, and on the other hand $q \in F$. In other words, $S_n(x_{k(n)})$ converges weakly to z and $\{x_n\}$ is almost weakly convergent to z. An appeal to the sufficiency part of [9, Theorem 7] completes the proof of (a). In order to prove (b), we denote $\lim_{n\to\infty} (x_n, x_{n+i})$ by g_i and recall that $S_n(g_0) \to | z |^2$. Since

$$| S_n(x_{k(n)})|^2 \leq (2/n^2) \sum_{0 \leq p \leq q \leq n-1} (x_{k(n)+p}, x_{k(n)+q})$$

$$\leq (2/n^2) \sum_{0 \leq p \leq q \leq n-1} (g_{q-p} + \epsilon_p)$$

with $\epsilon_p \to 0$, $\limsup_{n\to\infty} | S_n(x_{k(n)})| \leq | z |$, and $S_n(x_{k(n)}) \to z$. Thus $\{x_n\}$ is almost convergent to z. The proof is now finished by extending [9, Theorem 7] to Banach spaces.

This theorem improves upon the results of [6]. In particular, (as has also been shown by Bruck [7] who used a different argument), [6, Theorem 2] remains true even if $\{a_{n,k}\}$ is not assumed to be "proper." Strong convergence also occurs when T is compact (or more generally, condensing). It is not known if Theorem 1 is valid outside Hilbert space (even if one considers only the Cesàro method). See [5] for a different extension of [1] to l^p, $1 < p < \infty$.

For each positive s let $K(s, t)$: $[0, \infty) \to (-\infty, \infty)$ be of bounded variation in $[0, \infty)$, and denote its total variation by $V(s)$. We say that the kernel K is strongly regular if $\int_0^\infty | K(s, t)| \, dt$ is bounded in s, $\lim_{s\to\infty} \int_0^\infty K(s, t) \, dt = 1$, $\lim_{s\to\infty} \int_0^T K(s, t) \, dt = 0$ for all finite T, and $\lim_{s\to\infty} V(s) = 0$. The following result extends [2, Theorem 1] and [10, Theorem 2.4].

THEOREM 2. *Let H be a real Hilbert space, C a closed convex subset of H, $S : [0, \infty) \times C \to C$ a nonexpansive semigroup with a fixed point, and $x \in C$. Let K be strongly regular and*

$$R(s, x) = \int_0^\infty K(s, t) \, S(t, x) \, dt.$$

(a) *$R(s, x)$ converges weakly to a fixed point of S that is the asymptotic center of $S(t, x)$.*

(b) *If, in addition, $\lim_{t\to\infty} (S(t, x), S(t + r, x))$ exists uniformly in $r \geq 0$ (for example if S is odd), then $R(s, x)$ converges strongly.*

Proof. We use the proof of [10, Theorem 2.4]. Let $s_n \to \infty$, $h > 0$, and $a_{n,k} = \int_{kh}^{(k+1)h} K(s_n, t)\, dt$. Since $\{a_{n,k}\}$ is strongly regular, we can apply Theorem 1 to $S(h, \cdot)$: $C \to C$ and obtain the existence of a limit $q(h)$. $\{q(h): h > 0\}$ is Cauchy and converges strongly to q. It follows that if (a) ((b)) holds, then the weak (strong) $\lim_{n\to\infty} R(s_n, x) = q$. Hence the result.

Remark. For linear T and S, the Yosida mean ergodic theorem yields strong convergence of $\{y_n\}$ and $R(s, x)$ in any reflexive Banach space.

Note added in proof. Refinements of recent ideas of J.-B. Baillon ("Comportement asymptotique des itérés de contractions non linéaires dans les espaces L^p," *C.R. Acad. Sci. Paris* **286** (1978), 157–159) lead to the following partial extensions of Theorems 1 and 2 to Banach spaces (see R. E. Bruck, "A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces," to appear, and my paper entitled "Weak convergence theorems for nonexpansive mappings in Banach spaces," *J. Math. Anal. Appl.*, to appear):

THEOREM 3. Let C be a closed convex subset of a uniformly convex Banach space with a Fréchet differentiable norm, $T: C \to C$ a nonexpansive mapping with a fixed point, and $x \in C$. If the matrix $\{a_{n,k}\}$ is strongly regular and $y_n = \sum_{k=0}^{\infty} a_{n,k} T^k x$, then $\{y_n\}$ converges weakly to a fixed point of T.

THEOREM 4. Let C be a closed convex subset of a uniformly convex Banach space with a Fréchet differentiable norm, $S: [0, \infty) xC \to C$ a nonexpansive semigroup with a fixed point, and $x \in C$. If the kernel K is strongly regular and $R(s, x) = \int_0^\infty K(s, t)\, S(t, x)\, dt$, then $R(s, x)$ converges weakly to a fixed point of S.

It follows that $\{T^n x\}$ $(S(t, x))$ converges weakly to a fixed point of $T(S)$ if and only if the weak $\lim_{n\to\infty} (T^n x - T^{n+1} x) = 0$ (the weak $\lim_{t\to\infty} (S(t+h, x) - S(t, x)) = 0$ for all $h > 0$).

It remains an open problem whether in the setting of Theorems 3 and 4 strong convergence occurs for odd T and S. It is known, however that if the Banach space is uniformly convex, $C = -C$, and $T(S)$ is odd, then $\{T^n x\}(S(t, x))$ converges strongly to a fixed point of $T(S)$ if and only if the strong $\lim_{n\to\infty}(T^n x - T^{n+1} x) = 0$ (the strong $\lim_{t\to\infty} S(t+h, x) - S(t, x)) = 0$ for all $h > 0$). See Theorems 1.1 and 4.1 in J.-B. Baillon, R. E. Bruck and S. Reich, "On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces," *Houston J. Math.* **4** (1978), 1–9.

REFERENCES

1. J.-B. BAILLON, Un théorème de type ergodique pour les contractions nonlinéaires dans un espace de Hilbert, *C. R. Acad. Sci. Paris* **280** (1975), 1511–1514.
2. J.-B. BAILLON, Quelques propriétés de convergence asymptotique pour les semi-groupes de contractions impaires, *C. R. Acad. Sci. Paris* **283** (1976), 75–78.
3. J. B. BAILLON, Quelques propriétés de convergence asymptotique pour les contractions impaires. *C. R. Acad. Sci. Paris* **283** (1976), 587–590.
4. J.-B. BAILLON AND H. BRÉZIS, Une remarque sur le comportement asymptotique des semigroupes non linéaires, *Houston J. Math.* **2** (1976), 5–7.
5. B. BEAUZAMY AND P. ENFLO, Théorèmes de point fixe et d'approximation, to appear.
6. H. BRÉZIS AND F. E. BROWDER, Nonlinear ergodic theorems, *Bull. Amer. Math. Soc.* **82** (1976), 959–961.

7. R. E. BRUCK, JR., On the ergodic convergence of iterates of a nonexpansive mapping in Hilbert space and the structure of the weak ω-limit set, to appear.
8. M. EDELSTEIN, The construction of an asymptotic center with a fixed point property, *Bull. Amer. Math. Soc.* **78** (1972), 206–208.
9. G. G. LORENTZ, A contribution to the theory of divergent sequences, *Acta Math.* **80** (1948), 167–190.
10. S. REICH, Nonlinear evolution equations and nonlinear ergodic theorems, *J. Nonlinear Analysis* **1** (1977), 319–330.